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Iterative methods based on domain decomposition, with particular reference to problems of machining distortion

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Abstract

We consider some iterative methods for problems of elasticity theory based on the idea of substructuring. Suppose that the given complicated domain can be decomposed into two or more simple subdomains for which solutions are available. In contrast to customary methods of substructuring, we allow subdomains to overlap, or one to be inside another. The given domain can be formed in a rather flexible way as a union, product, or relative complement (difference) of subdomains. We study several iterative schemes of patching solutions for subdomains, to give the solution of the required problem. Convergence is established by presenting the operator of an iterative procedure as a product of two or more projectors. We then apply one of the algorithms for a problem of machining distortion as a result of residual stresses.

1. Introduction

We seek the solution to the system of linear elasticity equations

$$(\lambda + \mu) \text{ grad div } \mathbf{u} + \mu \Delta \mathbf{u} = \mathbf{q} \tag{1}$$

for a domain $G = G_1 + G_2$, as shown in Fig. 1a. In equation (1), $\mathbf{u}$ is the unknown vector of displacements, λ and μ are Lamé's constants and the vector function $\mathbf{q}$ represents mechanical or thermal loading. Let us consider the following algorithm. Solve equation (1) for the domain G_1, assuming arbitrary displacements $\mathbf{u}^{(0)}$ at the surface S_1', and with the given boundary conditions at S_1. The solution to this problem gives the vector $\mathbf{u}^{(0)}$ everywhere in the domain G_1, including S_2'. Take these as boundary values and solve (1) for the domain G_2, with conditions on S_2 as they are given in the problem. This solution provides us with boundary values $\mathbf{u}^{(1)}$ on S_1', which can be used for the next iteration in G_1. Then the procedure is repeated. This algorithm was initially suggested by Shwarz as a proof of existence of the solution to the harmonic equation for a multiply-connected domain [1]. It is now recognized [2] as a numerical procedure for a boundary-value problem, if the domain is constructed from simple subdomains, for which solutions are available. The convergence of the method, when displacements on the boundary are given, has been proven by S.L. Sobolev [3]. The method also has been applied to problems of elasticity theory with a given traction vector on the boundary [4–6]. In the latter version, continuity of the traction vector (not displacement) across surfaces S_1' and S_2' is preserved. It was indicated in both cases that a modification of the method can be used for a domain which is an intersection of simple subdomains. Applications of the method to problems of elastic stability were made in [7]. In the present paper, we show that both versions of the method, [3] and [4–6], converge for any customary combination of boundary

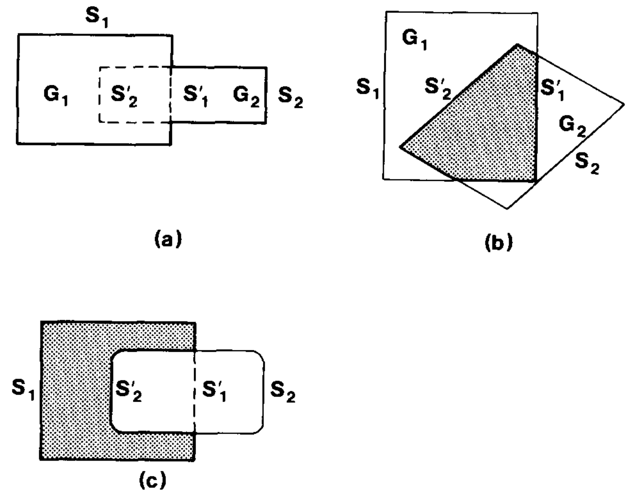

Fig. 1. Decomposition of the given domain into subdomains.

conditions. For example, on some portions of the boundary displacements may be specified, while some other portions are elastically supported, or have specified tractions. We also consider some new modifications of the procedure. Convergence in energy is established by presenting the operators of the procedure as a product of two or more operators of orthogonal projections (projectors).

We also demonstrate one of the algorithms on an example, related to a problem of machining distortion.

2. Algorithm of [3] with generalized boundary conditions

2.1. Preliminary remarks

Suppose that on some portion of the boundary, A_1, displacements are given,

$$u_i = f_i, \quad i = 1, 2, 3, \tag{2}$$

while on another portion of the boundary, A_2, tractions are specified,

$$\sigma_{ij} n_j = t_i, \quad i, j = 1, 2, 3, \tag{3}$$

and the remaining portion of the surface $S_1 \cup S_2$, A_3, is elastically supported,

$$\sigma_{ij} n_j + D_{ij} u_j = 0, \quad i, j = 1, 2, 3. \tag{4}$$

In these equations, σ_{ij} is the stress tensor, n_j the unit vector normal to the surface, and D_{ij} the nonnegative stiffness matrix of the elastic support.

We will use the known result that in linear problems the specific form of right-hand sides of equations does not influence the convergence of an iterative scheme. Therefore, it is possible to assume $\mathbf{q}$, f_i and t_i in equations (1–3) equal to zero. If started with some initial perturbation for the homogeneous problem, the process tends towards zero, then for the given non-homogeneous problem the process converges to the required solution. We study convergence, using the following scalar product and norm:

$$(\mathbf{u}, \mathbf{v}) = \int_G \sigma_{ij}(\mathbf{u}) e_{ij}(\mathbf{v}) \,\mathrm{d}G + \int_{A_3} u_i D_{ik} v_k \,\mathrm{d}S$$

$$= \int_G \sigma_{ij}(\mathbf{v}) e_{ij}(\mathbf{u}) \,\mathrm{d}G + \int_{A_3} v_i D_{ik} u_k \,\mathrm{d}S, \quad i,\ j,\ k = 1, 2, 3, \tag{5}$$

$$|\mathbf{u}|^2 = (\mathbf{u}, \mathbf{u}). \tag{6}$$

We can see that the norm defined by (5,6) is the doubled potential energy of elastic deformation of the system, including potential energy of elastic supports on A_3. Symmetry of expression (5) follows from reciprocity of energy and can be verified directly by integration by parts.

2.2. Union of subdomains

The procedure suggested in [3].can be represented as the product of two operators P_1 and P_2, which are defined as follows.

For operator P_1:
$P_1\mathbf{u} = \mathbf{u}$ in $G_2 - G_1$, $P_1\mathbf{u}^- = \mathbf{u}^+$ on S_1', $P_1\mathbf{u}$ satisfies equation (1) in G_1, and boundary conditions (2-4) on S_1. Superscripts "$-$" and "$+$" respectively denote values on the left- and right-hand sides of the surface.

For operator P_2:
$P_2\mathbf{u} = \mathbf{u}$ in $G_1 - G_2$, $P_2\mathbf{u}^- = \mathbf{u}^+$ on S_2', $P_2\mathbf{u}$ satisfies equation (1) in G_2, and boundary conditions (2-4) on S_2. Then the iterative procedure can be described as

$$\mathbf{u}^{(n+1)} = P\mathbf{u}^{(n)} = P_2 P_1 \mathbf{u}^{(n)}. \tag{7}$$

THEOREM 1. *Procedure (7) is convergent in the energy norm (6).* The burden of proof is on the following lemma.

LEMMA 1. *Operators P_1 and P_2 are projectors in norm (6).*

Proof: Consider a set of vector functions $\{\mathbf{g}_i\}$, generated by operator P_1. It is sufficient to show that for any $\mathbf{u}$ and $\mathbf{g}$

$$(\mathbf{u} - P_1\mathbf{u}, \mathbf{g}_i) = 0. \tag{8}$$

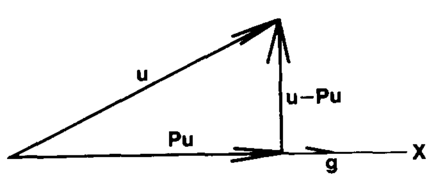

Fig. 2. Geometrical illustration of condition (8).

This condition is geometrically illustrated in Fig. 2 which shows that an operator P is a projector onto axis x if and only if $\mathbf{u} - P\mathbf{u}$ is perpendicular to a $\mathbf{g}$ directed along x.

Using integration by parts, the scalar product (5) can be rewritten as

$$(\mathbf{u}, \mathbf{v}) = \int_G \sigma_{ij,j}(\mathbf{u}) v_i \, \mathrm{d}G + \int_{A_1} \sigma_{ij}(\mathbf{u}) n_j v_i \, \mathrm{d}S + \int_{A_2} \sigma_{ij}(\mathbf{u}) n_j v_i \, \mathrm{d}S$$

$$+ \int_{A_3} \sigma_{ij}(\mathbf{u}) n_j v_i \, \mathrm{d}S + \int_{S_1'} \left(\sigma_{ij}(\mathbf{u})^+ - \sigma_{ij}(\mathbf{u})^-\right) n_j v_i \, \mathrm{d}S$$

$$+ \int_{S_2'} \left(\sigma_{ij}(\mathbf{u})^+ - \sigma_{ij}(\mathbf{u})^-\right) n_j v_i \, \mathrm{d}S + \int_{A_3} u_i D_{ij} v_j \, \mathrm{d}S. \tag{9}$$

There are two unit normal vectors on $S_1' \cup S_2'$:

$$n_j^+ = -n_j^-. \tag{10}$$

In (9) and below we use only one of them, n_j^+, and omit the superscript. The integral in the domain G vanishes since equilibrium equations

$$\sigma_{ij,j} = 0 \tag{11}$$

are satisfied for each subdomain. The integrals along surfaces A_1 and A_2 equal zero because of homogeneous boundary conditions corresponding to (2) and (3). The two integrals along A_3 cancel, as can be seen from (4). Then expression (9) becomes

$$(\mathbf{u}, \mathbf{v}) = \int_{S_1'} \left(\sigma_{ij}(\mathbf{u})^+ - \sigma_{ij}(\mathbf{u})^-\right) n_j v_i \, \mathrm{d}S + \int_{S_2'} \left(\sigma_{ij}(\mathbf{u}) - \sigma_{ij}(\mathbf{u})^-\right) n_j v_i \, \mathrm{d}S. \tag{12}$$

This expression can be used to verify (8), giving

$$(\mathbf{u} - P_1\mathbf{u}, \mathbf{g}) = \int_{S_1'} \left(\sigma_{ij}(\mathbf{g})^+ - \sigma_{ij}(\mathbf{g})^-\right) n_j \left[u_i - (P_1\mathbf{u})_i\right] \mathrm{d}S + \int_{S_2'} \left(\sigma_{ij}(\mathbf{g})^+\right.$$

$$\left. - \left(\sigma_{ij}(\mathbf{g})^-\right) n_j \left[u_i - (P_i\mathbf{u})_i\right] \mathrm{d}S. \tag{13}$$

The first integral in (13) is equal to zero because, by the definition of $P_1 u_i$, $(P_1\mathbf{u})_i = 0$ on S_1'. The second integral is equal to zero, since $\mathbf{g}$ is continuous on S_2' together with all derivatives and therefore, $\sigma_{ij}(\mathbf{g})^+ - \sigma_{ij}(\mathbf{g})^- = 0$. Equation (8) is verified and P_1 is indeed a projector. The same statement is true with respect to operator P_2. Theorem 1 is now a consequence of properties of projectors [8].

2.3. *Product of subdomains*

We will now consider an algorithm for the intersection of the subdomains, as shown in Figure 1b. Let boundary conditions be of the form (2) on $S_1' + S_2'$. Start the solution of (1) for the domain G_1. We can take any combination of homogeneous conditions corresponding to (2-4) on S_1, which is convenient for calculations, since this boundary is obtained artificially by expanding the given domain and is not needed for the actual solution. The resulting vector function $\mathbf{u}^{(0)}$ most likely will not coincide with that given on S_2'. Take the difference (error) $f_i(S_2') - u_i^{(0)}(S_2')$ as the boundary value for the domain G_2 and solve equation (1) with $\mathbf{q} = 0$ and an arbitrary combination of homogeneous conditions (2-4) on S_2. This corrective solution causes conditions on S_2' to be satisfied, but there will be some error on S_1'. Take this error as the boundary condition for the problem in the domain G_1, etc. For the study of convergence we consider the homogeneous equations with some initial perturbations. We then notice that the consequent iterations may differ only in sign from those obtained in the previous section for $G_1 \cup G_2$, and therefore both versions are convergent.

2.4. *Relative complement of domains*

Let us consider now the domain, as shown in Fig. 1c with arbitrary combination of conditions (2-4) on S_1 and prescribed displacements on S_2'. Iterations are constructed as follows: solve equation (1) in G_1 with given boundary conditions on S_1 and displacements $u_i = f_i$ chosen arbitrarily on S_1'. Most likely, this solution will not give the needed displacements on S_2'. Take the error on S_2' as the boundary value for the homogeneous problem in the domain G_2 and obtain corrections for boundary conditions on S_1', etc. Again, we can take any combination of homogeneous conditions on S_2. For the corresponding homogeneous problem with initial perturbation the algorithm can be described as

$$\mathbf{u}^{(n+1)} = (I - P_2) P_1 \mathbf{u}^{(n)} \tag{14}$$

where I is the identity operator and P_1, P_2 were described before. Therefore, $I - P_2$ is a projector, and the process is convergent in energy.

3. Algorithm of [4-6] with generalized boundary conditions

The algorithms considered here are similar to those in the previous section. The difference is that now we preserve continuity of traction vector $\sigma_{ij} n_j$ on the surfaces $S_1' + S_2'$ rather than continuity of displacements. We will use notations Q_1 and Q_2 for the corresponding operators.

THEOREM 2. *Operators Q_1 and Q_2 are projectors, and therefore algorithm $\mathbf{u}^{(n+1)} = Q_2 Q_1 \mathbf{u}^{(n)}$ is convergent in energy.*

Proof: It is sufficient to show that

$$(\mathbf{u}, Q_1\mathbf{u}) = (Q_1\mathbf{u}, Q_1\mathbf{u}). \tag{15}$$

Using derivations similar to those of (9-11), and taking into account that the normal and tangential stresses are now continuous across $S_1' \cup S_2'$, we obtain instead of (13):

$$(\mathbf{u}, \mathbf{v}) = \int_{S_1' \cup S_2'} \sigma_{ij}(\mathbf{u}) n_j (v_j^+ - v_j^-) \, \mathrm{d}S. \tag{16}$$

Then

$$(Q_1\mathbf{u}, Q_1\mathbf{u}) = \int_{S_1'} \sigma_{ij}(Q_1\mathbf{u}) n_j \left[(Q_1\mathbf{u})_j^+ - (Q_1\mathbf{u})_j^- \right] \mathrm{d}S + \int_{S_2'} \sigma_{ij}(Q_1\mathbf{u}) n_j \left[(Q_1\mathbf{u})_j^+ - (Q_1\mathbf{u})_j^- \right] \mathrm{d}S. \tag{17}$$

By definition of the operator Q_1,

$$\sigma_{ij}(Q_1\mathbf{u})n_j = \sigma_{ij}(\mathbf{u})n_j \quad \text{on } S_1', \tag{18}$$

$$(Q_1\mathbf{u})_j^+ = (Q_1\mathbf{u})_j^- \quad \text{on } S_2'. \tag{19}$$

Therefore

$$(Q_1\mathbf{u}, Q_1\mathbf{u}) = \int_{S_1'} \sigma_{ij}(\mathbf{u})n_j\left[(Q_1\mathbf{u})_j^+ - (Q_1\mathbf{u})_j^-\right] \mathrm{d}S. \tag{20}$$

On the other hand

$$(\mathbf{u}, Q_1\mathbf{u}) = \int_{S_1'} \sigma_{ij}(\mathbf{u})n_j\left[(Q_1\mathbf{u})_j^+ - (Q_1\mathbf{u})_j^-\right] \mathrm{d}S + \int_{S_2'} \sigma_{ij}(\mathbf{u})n_j\left[(Q_1\mathbf{u})_j^+ - (Q_1\mathbf{u})_j^-\right] \mathrm{d}S. \tag{21}$$

The second integral in (21) vanishes because of (19). Then comparison (20) and (21) gives (15).

In a similar manner, convergent algorithms for domains $G_1 \cap G_2$ and $G_1 - G_2$ can be constructed.

4. Examples

4.1. Circular segment on a rigid foundation

To illustrate some features of the treated algorithms, we consider an example shown in Fig. 3a. We assume that the foundation is smooth (no tangential stresses), and that no separation is permitted. These boundary conditions can be achieved by presenting the problem as shown in

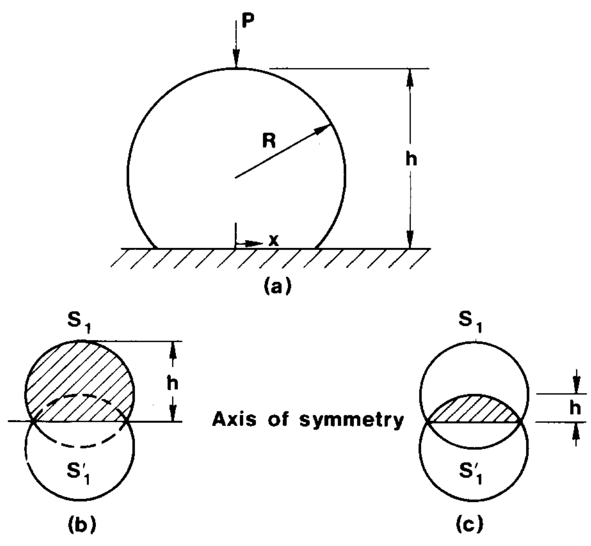

Fig. 3. Circle segment on a rigid smooth foundation. A. Given scheme. B. Decomposition into sum of subdomains. C. Decomposition into product of subdomains.

Table 1. Number of iterations for different overlapping in Fig. 3b (sum of the domains).

h/R in Fig. 3a	1.8	1.6	1.4	1.2
Number of iterations	15	7	4	2

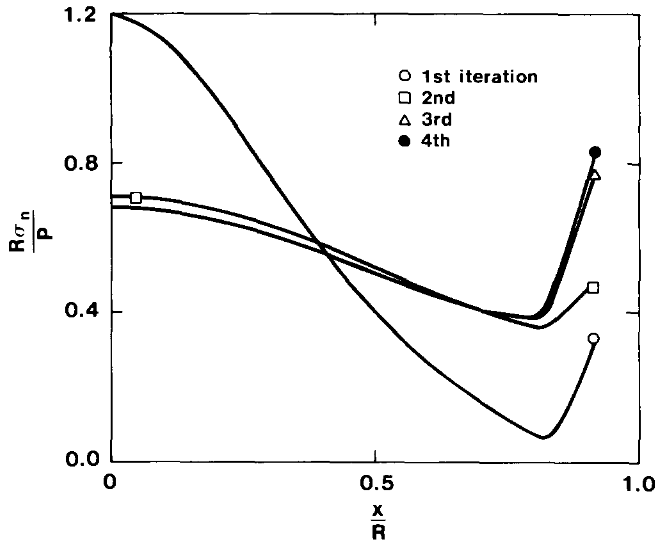

Fig. 4. Convergence of contact pressure for $h/R = 1.4$.

Fig. 3b, using symmetry. The problem is described in terms of a stress function governed by the biharmonic equation. For each subdomain, the solution is

$$f = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{n+2}\right) \cos n\theta \tag{22}$$

where coefficients A_n and B_n are chosen to satisfy the given boundary conditions. The procedure for each subdomain seems to be straightforward and we omit it. We start the iteration described in Section 3 with some arbitrary stress distribution on S_1', equilibrating the given force P. We chose to put the force P at the mid-point of S_1'. Iterations were aborted when corrections for stresses on S_2' in the current iteration were less then some prescribed fraction (1% in our analysis) of total stresses. The number of terms in (22) was taken as 100. Table 1 shows, as we should expect, that the convergence improves, as the overlapping portion of the subdomains increases. Figure 4 shows the convergence of the contact stress distribution for $h/R = 1.4$. Apparently, we cannot use the representation in Fig. 3b to obtain the solution for $h/R \leqslant 1$. But in this case, we can use the algorithm for the intersection of subdomains, as shown in Fig. 3c. Results for different overlapping are given in Table 2. Again, convergence improves, as overlapping of subdomains increases. The contact pressure at the mid-point is shown in Fig. 5 where the portion of the curve for $h/R < 1$ was obtained with the algorithm for

Table 2. Number of iterations for different overlapping in Fig. 3c (intersection of the domains)

h/R in Fig. 3a	0.2	0.4	0.6	0.8
Number of iterations	4	3	3	2

Table 3. Number of iterations for different overlapping in Fig. 6.

h/R in Fig. 6	0.25	0.5	0.75	1.0	1.25	1.5	1.75
Number of iterations	2	4	5	6	9	10	12

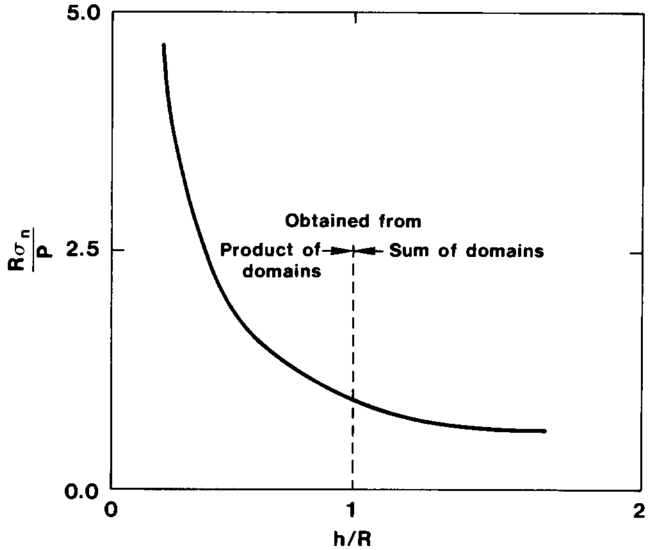

Fig. 5. Contact pressure of mid-point as a function of h/R.

the intersection of the subdomains, while for the portion $h/R > 1$ we used the algorithm for the unity of subdomains.

4.2. A problem of machining distortion

Suppose, residual stresses were introduced in a part in the process of manufacturing: quenching, for example. Then during machining some distortion of the part is observed. To visualize this, one notes that machining releases the residual tractions on the newly formed surfaces, which is mechanically equivalent to application of these reversed tractions to the remaining portion of the part. Machining is also used for experimental verification of analytically obtained residual stresses. Strain gauges are placed at selected points of the surface, and deformations are measured, while some portions of material are removed. These deformations are compared with those obtained on the basis of modeled residual stresses. Consider the following example. It was found [9] that the residual stresses in a long cylinder can be approximated by

$$\sigma_r = 872 \cos(\pi r/2R) \text{ N/cm}^2 \tag{23}$$

with other stresses given by

$$\sigma_\theta = \frac{\mathrm{d}}{\mathrm{d}r}(r\,\sigma_r), \tag{24}$$

$$\tau = 0. \tag{25}$$

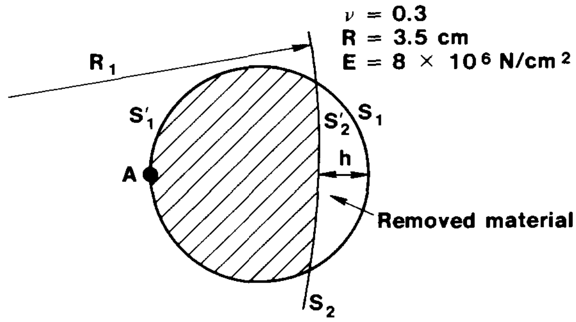

Fig. 6. Machining distortion of a cylinder with residual stresses.

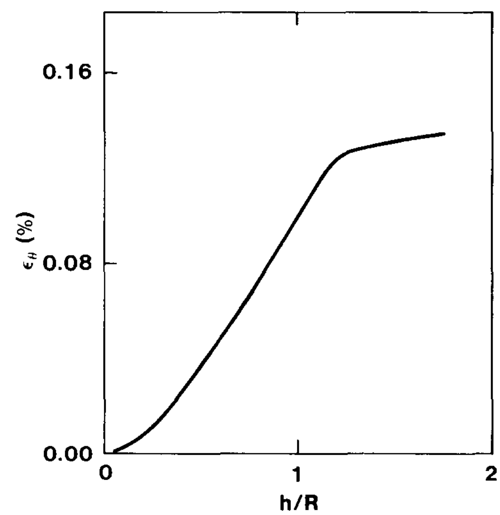

Fig. 7. Predicted deformation at point A of Fig. 6 during machining.

As illustrated in Fig. 6, we are interested in the development of deformation ϵ_θ at point A as we gradually remove material. The remaining domain can be represented as an intersection of two circles with the radius of one of them taken sufficiently large. Using the results (22) for a circle and the algorithm of Section 3 for the intersection of subdomains, the graph in Fig. 7 was obtained. Table 3 again shows that the algorithm used becomes more efficient as the overlapping area increases.

5. Conclusion

Efficiency of the considered algorithms depends on particular combination of boundary conditions and geometry. However, some general remarks can be made. It can be shown as in [7] that for the cases of union or product of subdomains, convergence should improve when overlapping increases. This conclusion is supported by the examples discussed in Section 4. For the case of difference of subdomains, the opposite is true.

We also note that the number of subdomains can be more than two, but the discussed algorithms are most efficient when the number of subdomains is small.

Comparing with other algorithms available, we emphasize that the suggested method is intended for use when the given domain is too complicated to be handled as the 'whole', while reasonably efficient solutions for subdomains are available. It is immaterial for the presented algorithm how specifically the equations for the subdomains will be solved. Suppose, for example, that the expressions

$$u(G_i) = \sum C_{ij} H_{ij}(x, y, z) \tag{26}$$

are available for each subdomain. Suppose also, that each term of (26) meets at least kinematic boundary conditions. Then the Ritz procedure can be employed for the subdomains, and the solutions can be patched by the suggested algorithm. This approach is advantageous when the expression of form (26) for the whole domain is unknown. Finally, it is not necessary that all subdomains have to be solved by the same method. The Ritz method can be used for the first subdomain, the Galerkin's for the second, finite-element method for the third, etc.

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